O-minimal homotopy and generalized (co)homology

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Abstract

This article extends the semialgebraic homotopy theory (developed by H. Delfs and M. Knebusch) to regular paracompact locally definable spaces and definable CW-complexes over a model R of an o-minimal (complete) theory T extending RCF, and even for weakly definable spaces, if T is a bounded theory. Corresponding generalized homology and cohomology theories for pointed weak polytopes essentially coincide with the known topological generalized theories if T is bounded.

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1 Introduction

In the 1980's, H. Delfs, M. Knebusch and others developed "semialgebraic topology" in locally semialgebraic and weakly semialgebraic spaces (see [5, 6, 7, 8, 17]). In the survey paper [18], M. Knebusch suggested that this theory may be generalized to the o-minimal context. This programme was undertaken first by A. Woerheide, who constructed o-minimal singular homology theory in [25], and later by M. Edmundo, who developed and applied singular homology theory and cohomology theory over o-minimal structures (see for example [11]). For homotopy theory, A. Berarducci and M. Otero worked with the o-minimal fundamental group and transfer methods in o-minimal geometry ([3, 4]). During the period of writing of this paper, several authors wrote about different types of homology and cohomology (see [12, 13], for example).

I show that the homotopy theory, developed in the semialgebraic case by H. Delfs and M. Knebusch in [8, 17], extends to regular paracompact locally definable spaces and definable CW-complexes over models R of an o-minimal (complete) theory T extending RCF, and even for arbitrary weakly definable spaces over R, if T is a bounded theory (see below for the definition). A consequence of this is that the generalized homology and cohomology theories for so called pointed weak polytopes may be defined and, if T is bounded, are essentially the same as their topological counterparts.

I re-develop the theory of H. Delfs and M. Knebusch for spaces over o-minimal expansions of real closed fields. Most of the facts can be proved as in [8] or [17], changing the word "semialgebraic" into the word "definable".

The categories of locally and weakly definable spaces over o-minimal expansions of real closed fields, introduced here, with their subspaces (locally definable subsets and weakly definable subsets) are far generalizations of *analytic-geometric categories* of van den Dries and Miller [9]. In particular paracompact locally definable manifolds are generalizations of both definable manifolds over o-minimal expansions of fields and real analytic manifolds.

For basic properties of o-minimal structures, see the book [10] and the survey paper [9]. **Assume that R is an o-minimal expansion of a real closed field.**

2 Spaces over o-minimal structures

As o-minimal structures have natural topology, it is quite self-explaining that (algebraic) topology for such structures should be developed. (This paper deals only with the case of o-minimal expansions of fields.) Unfortunately, there are obstacles to the above when one is doing traditional topology: if R is not (an expansion of) the (ordered) field of real numbers $\mathbb{R}$, then R is not locally compact and is totally disconnected. Moreover, not every family of open definable sets has a definable union.

A good idea to overcome that in the case of o-minimal pure (ordered) fields was given by H. Delfs and M. Knebusch in [8]: it is the concept of a generalized topological space. This idea serves well also in our setting.

A **generalized topological space** is a set M together with a family of subsets $\overset{\circ}{\mathcal{T}}(M)$ of M, called **open sets**, and a family of open families Cov_M, called **admissible (open coverings)**, such that:

(A1) $\emptyset, M \in \overset{\circ}{\mathcal{T}}(M)$ (the empty set and the whole space are are open),

(A2) if $U_1, U_2 \in \overset{\circ}{\mathcal{T}}(M)$ then $U_1 \cup U_2, U_1 \cap U_2 \in \overset{\circ}{\mathcal{T}}(M)$ (finite unions and finite intersections of open sets are open),

(A3) if $\{U_i\}_{i\in I} \subseteq \overset{\circ}{\mathcal{T}}(M)$ and I is finite, then $\{U_i\}_{i\in I} \in \mathrm{Cov}_M$ (finite families of open sets are admissible),

(A4) if $\{U_i\}_{i\in I} \in \mathrm{Cov}_M$ then $\bigcup_{i\in I} U_i \in \overset{\circ}{\mathcal{T}}(M)$ (the union of an admissible family is open),

(A5) if $\{U_i\}_{i\in I} \in \mathrm{Cov}_M$, $V \subseteq \bigcup_{i\in I} U_i$, and $V \in \overset{\circ}{\mathcal{T}}(M)$, then $\{V \cap U_i\}_{i\in I} \in \mathrm{Cov}_M$ (the traces of an admissible family on an open subset of the union of the family form an admissible family),

(A6) if $\{U_i\}_{i\in I} \in \mathrm{Cov}_M$ and for each $i \in I$ there is $\{V_{ij}\}_{j\in J_i} \in \mathrm{Cov}_M$ such that $\bigcup_{j\in J_i} V_{ij} = U_i$, then $\{V_{ij}\}_{\substack{i\in I\\ j\in J_i}} \in \mathrm{Cov}_M$ (members of all admissible coverings of members of an admissible family form together an admissible family),

(A7) if $\{U_i\}_{i\in I} \subseteq \overset{\circ}{\mathcal{T}}(M)$, $\{V_j\}_{j\in J} \in \mathrm{Cov}_M$, $\bigcup_{j\in J} V_j = \bigcup_{i\in I} U_i$, and $\forall j \in J\, \exists i \in I :$ $V_j \subseteq U_i$, then $\{U_i\}_{i\in I} \in \mathrm{Cov}_M$ (a coarsening, with the same union, of an admissible family is admissible),

(A8) if $\{U_i\}_{i\in I} \in \mathrm{Cov}_M$, $V \subseteq \bigcup_{i\in I} U_i$ and $V \cap U_i \in \overset{\circ}{\mathcal{T}}(M)$ for each i, then $V \in \overset{\circ}{\mathcal{T}}(M)$ (if a subset of the union of an admissible family has open traces with members of the family, then the subset is open).

Generalized topological spaces may be identified with certain Grothendieck sites, where the underlying category is a full, closed on finite (in particular: empty) products and coproducts subcategory of the category of subsets $\mathcal{P}(M)$ of a given set M with inclusions as morphisms, and the Grothendieck topology is subcanonical, contains all finite jointly surjective families and satisfies some regularity condition. (See [19] for the

definition of a Grothendieck site. Considering such an identification we should remember the ambient category $\mathcal{P}(M)$.) More precisely: the axioms (A1), (A2) and (A3) form a stronger version of the identity axiom of the Grothendieck topology. It is natural, since in model theory and in geometry we love finite unions and finite intersections. The axiom (A4) may be called *co-subcanonicality*. Together with subcanonicality, it ensures that admissible coverings are coverings in the traditional sense (which is imposed by the notation of [8]). The next are: (A5) the stability axiom of the Grothendieck topology, followed by the transitivity axiom (A6). Finally, (A7) is the saturation property of the Grothendieck topology (usually the Grothendieck topology of a site is required to be saturated), and the last axiom (A8) may be called the *regularity axiom*. Both saturation and regularity have a smoothing character. Saturation may be achieved by modifying any generalized topological space, and regularity by modifying a locally definable space (see I.1, page 3 and 9 of [8]).

A **strictly continuous mapping** between generalized topological spaces is such a mapping that the preimage of an open set is open and the preimage of an (open) admissible covering is admissible. (So strictly continuous mappings may be seen as morphisms of sites in this context.) Inductive limits exist in the category **GTS** of generalized topological spaces and their strictly continuous mappings (see I.2 in [8]).

Generalized topological spaces help to introduce further notions of interest that are generalizations of corresponding semialgebraic notions.

A **function sheaf of rings over** R on a generalized topological space M is a sheaf F of rings on M (here the sheaf property is assumed only for admissible coverings) such that for each U open in M the ring $F(U)$ is a subring of the ring of all functions from U into R, and the restrictions of the sheaf are the set-theoretical restrictions of mappings. A **function ringed space over** R is a pair (M, O_M), where M is a generalized topological space and O_M is a function sheaf of rings over R. We will say about **spaces** (over R) for short. An **open subspace** of a space over R is an open subset of its generalized topological space together with the function sheaf of the space restricted to this open set. A **morphism** $f : (M, O_M) \to (N, O_N)$ of function ringed spaces over R is a strictly continuous mapping $f : M \to N$ such that for each open subset V of N the set-theoretical substitution $h \mapsto h \circ f$ gives a morphism of rings $f^\#_V : O_N(V) \to O_M(f^{-1}(V))$. (We could say that $f^\# : O_N \to f_* O_M$ is the morphism of sheaves of rings on N over R induced by f. However, if we define for function sheaves

$$(f_* O_M)(V) = \{h : V \to R |\, h \circ f \in O_M(f^{-1}(V))\},$$

then each $f^\#_V : O_N(V) \to f_* O_M(V)$ becomes just an inclusion.) Inductive limits exist in the category $\mathbf{Space}(R)$ of spaces over any R and their morphisms (cf. I.2). Notice that our category of spaces over R, being a generalization of the category of spaces from [17], is nicer that the one considered in [8].

Fundamental Example 1. *Each definable subset D of R^n has a natural structure of a function ringed space over R. Its open sets in the sense of the generalized topology are (relatively) open definable subsets, admissible coverings are such open coverings that already finitely many open sets cover the union, and on each open definable subset we take the ring $\mathcal{DC}_D$ of all continuous definable R-valued functions.* Definable sets will be identified with such function ringed spaces. *Notice that the topological closure of a definable set is definable, so the topological closure operator restricted to the class of definable subsets*

of a definable set D can be treated as the closure operator in the generalized topological sense.

An **affine definable space** over R is a space over R isomorphic to a definable subset of some R^n. (Notice that morphisms of affine definable spaces are given by continuous definable maps between definable subsets of affine spaces.)

The next example shows that it is important to consider affine definable spaces as definable sets "embedded" into their ambient affine spaces.

Example 2 ("Bad boy"). *The semialgebraic space $\mathbb{R}/\mathbb{Z}$ may be identified with the set $S^1 \subseteq \mathbb{R}^2$, where the members of the structure sheaf are (periodic) continuous semialgebraic functions of the angle θ. This space is not semialgebraically isomorphic to the embedded circle S^1 of the affine space R^2. Still its affine model is the usual circle: there exists an isomorphism of (semialgebraic) spaces over $\mathbb{R}$ (whose formula is not semialgebraic) transforming the "non-embedded circle" into the "embedded circle".*

A **definable space** over R is a space over R that has a finite open covering by affine definable spaces. (Definable spaces were introduced by van den Dries in [10]. They have clear notions of a definable subset and of an open subset. The definable subsets of a definable space form a Boolean algebra generated by the open definable subsets; "definable" here means "constructible from the generalized topology"). A **locally definable space** over R is a space over R that has an admissible covering by affine definable open subspaces. (So definable spaces are examples of locally definable spaces.) Each locally definable space is an inductive limit of a directed system of definable spaces in the category of spaces over a given R (cf. I.2.3). **Morphisms** of affine definable spaces, definable spaces and locally definable spaces over R are their morphisms as spaces over R. So affine definable spaces, definable spaces and locally definable spaces form full subcategories $\mathbf{ADS}(R)$, $\mathbf{DS}(R)$, and $\mathbf{LDS}(R)$ of the category $\mathbf{Space}(R)$ of spaces (over R).

A **locally definable subset** of a locally definable space is a subset having definable intersections with all open definable subspaces. Such subsets are also considered as **subspaces**, the locally definable space of such a set is formed as inductive limits of definable subspaces of open definable spaces forming the ambient space (cf. I.3, p. 28 in [8].) A locally definable subset of a locally definable space is called **definable** if as a subspace it is a definable space. (The definable subsets of a definable space are exactly the definable subsets of these spaces as locally definable ones.)

On locally definable spaces we often introduce a topology in the traditional sense, called the **strong topology**, taking the open sets from the generalized topology as the basis of the topology. Nevertheless, we will usually work in the generalized topology. This allows, in many cases, to omit the word "definably" applied to topological notions (as in "definably connected"). On a definable space the generalized topology generates both the strong topology and the definable (i. e. constructible) subsets. Similarly, the locally definable subsets of a locally definable space are exactly the sets "locally constructible" from the generalized topology, where "locally" means "when restricted to an open definable subspace". The closure operator of the strong topology restricted to the class of locally definable subsets may be treated as the closure operator of the generalized topology.

Example 3. *Take the union of real line open intervals $(-\infty, n)$ over all natural n as a space, which means that this family is assumed to be admissible. Then this space is*

definable "on the left-hand side", but only locally definable "on the right-hand side". The definable subsets are the finite unions of intervals (of any kind) that are bounded from above. The locally definable subsets are locally finite unions of intervals that have only finitely many connected components on the negative half-line. This space will be called $\mathbb{R}_{loc,+}$. Analogously we define the space $\mathbb{R}_{loc,-}$ requiring the family $(-n, +\infty)$, for $n \in \mathbb{N}$, to be admissible.

Example 4. *Any "direct topological sum" of definable spaces (in the category of spaces over a given R) is a locally definable space (cf. I.2.4 in [8]).*

We call a subset K of a generalized topological space M **small** if for each admissible covering of any open U, the set $K \cap U$ has a finite subcovering. Just from the definitions, we get

Facts 5. *Each definable space is small. Each subset of a definable space is also small. Every small open subspace of a locally definable space is definable. Each small set of a locally definable space is contained in a small open set. In particular "small open" means exactly "definable open".*

Any locally definable space is (topologically) Hausdorff iff it is **Hausdorff** in the generalized topological sense. Similarly, a locally definable space is (topologically) regular iff it is **regular** in the generalized topological sense: any single point, always closed, and any closed subspace (not containing the point) can be separated by disjoint open subspaces.

Clearly, each affine definable space is regular. Of great importance for the theory od definable spaces is the following

Theorem 6 (Robson[22], van den Dries[10])**.** *Each regular definable space is affine.*

Remark 7. *Even if we define locally definable spaces with the use of structure sheaves, a locally definable space is determined by its generalized topology (when we assume silently that the structure of each affine subspace is understood) since it has an admissible covering of regular small (= affine) open subspaces. The main purpose of introducing function ringed spaces was to define morphisms.*

To determine a locally definable space, it is enough to set one admissible covering by definable open subspaces (the structure of a definable open space is silently understood as in the case of affine definable space). The practical way of denoting a locally definable space is to write it as the union of its admissible covering by open definable (often affine) subspaces, not just the union set (even if considered with a topology).

The author considers an attempt to encode the generalized topology under the notion of "equivalent atlases" a little bit risky. For example, we have the following fact about locally definable spaces over any archimedean R :

$$(0,1)_{loc} \stackrel{\text{def.}}{=} \bigcup_{n=3}^{\infty} (\frac{1}{n}, 1-\frac{1}{n}) \neq \bigcup_{n=3}^{\infty} (\frac{1}{n}, 1-\frac{1}{n}) \cup (0,1) = (0,1).$$

We would have two atlases that combine to a third atlas, but it does not imply that the structures of locally definable spaces are the same. The combined atlas would be equivalent to one of the initial atlases but not to the second.

Here the notation with the union means that the space we get is the inductive limit (in the category of spaces) of a system of definable open subspaces, partially ordered by inclusion, that are finite unions of the definable sets under the union. Sometimes taking finite unions is not necessary.

(In their recent paper [2], E. Baro and M. Otero introduce a locally definable space just as a particular atlas and quite early overlook the generalized topology, considering only the strong topology. This is motivated by their basic interest in section 3.2 in subsets of R^n, for sufficiently saturated R, as locally definable spaces introduced in their Example 3.1. The reader should be warned that in general a locally definable space highly depends on the choice of an atlas (corresponding here to an admissible covering by affine open subspaces) even if the topology of a space is fixed. Moreover, the definition and use of their "locally finite generalized simplicial complexes" and "ld-homeomorphisms" are not clear.)

We say that an object M of $\mathbf{LDS}(R)$ **comes from** R^k if the underlying topological space of M coincides with that of R^k and at each point of R^k the germs of the locally definable subsets of M and of the definable subsets of R^k form equal families.

Theorem 8. *For each o-minimal expansion $\mathbb{R}_{\mathcal{S}}$ of the field of real numbers $\mathbb{R}$:*

1) there are exactly four different objects of $\mathbf{LDS}(\mathbb{R}_{\mathcal{S}})$ that come from $\mathbb{R}^1_{\mathcal{S}}$, namely: $\mathbb{R}_{\mathcal{S}}$, $(\mathbb{R}_{\mathcal{S}})_{loc}$, $(\mathbb{R}_{\mathcal{S}})_{loc,+}$, $(\mathbb{R}_{\mathcal{S}})_{loc,-}$.

2) there are uncountably many different objects of $\mathbf{LDS}(\mathbb{R}_{\mathcal{S}})$ that come from $\mathbb{R}^2_{\mathcal{S}}$.

Proof. 1) Assume M is an object of $\mathbf{LDS}(\mathbb{R}_{\mathcal{S}})$ coming from $\mathbb{R}^1_{\mathcal{S}}$. Each finite union of bounded intervals is a definable subset of M. Each of the open intervals $(-\infty, 0)$ and $(0, +\infty)$ either is definable or not. This makes four cases. If one of these intervals is definable then its structure as a space is uniquely determined. If it is not definable, then we can cover it by a countable, locally finite (in the traditional sense, hence admissible) family of bounded open intervals, so again its structure of a locally definable space is uniquely determined. These four cases are realized by $\mathbb{R}_{\mathcal{S}}$, $(\mathbb{R}_{\mathcal{S}})_{loc}$, $(\mathbb{R}_{\mathcal{S}})_{loc,+}$, $(\mathbb{R}_{\mathcal{S}})_{loc,-}$.

2) Choose a slope $a \in \mathbb{R}$ and consider the space

$$M_a = \bigcup_{n \in \mathbb{N}} \{(x, y) \in \mathbb{R}^2 :\ y < ax + n\}.$$

All M_a, for $a \in \mathbb{R}$, are different objects of $\mathbf{LDS}(\mathbb{R}_{\mathcal{S}})$. □

Theorem 9. *For any o-minimal expansion R of a field not isomorphic to $\mathbb{R}$ there are already uncountably many different objects of $\mathbf{LDS}(R)$ that come from the line R^1.*

Proof. Case 1: R contains $\mathbb{R}$.

The set of galaxies of R is uncountable. For any galaxy G of R take $x \in G$ and consider the space N_G defined as the disjoint generalized topological union of all the galaxies $G' > G$, treated each one as a locally definable space (see Remark 12), and the space

$$M_G = \bigcup_{n \in \mathbb{N}} (-\infty, x + n),$$

which is the union of all galaxies $G'' < G$ "localized"(see below) only at the end of G.

Case 2: R does not contain $\mathbb{R}$.

The field $R \cap \mathbb{R}$ has uncountably many irrational cuts, determined by elements $r \in \mathbb{R} \setminus R$. For each such r, consider the space

$$M_r = \bigcup_{s<r}(-\infty, s) \cup \bigcup_{s>r}(s, +\infty)$$

consisting of two connected components given by conditions $x > r$ and $x < r$. □

As to the more general sets that will be called local subsets, called elsewhere "locally definable" (a subset Y of a space X is a **local subset** if for each point y only of Y the germ of Y at y is definable), they can be given of a locally definable space structure, but their properties are not nice: they are closed only on finite intersection, and are not closed under complement or union. The locally definable space on such Y is given as the following union of the admissible covering

$$Y = \bigcup \{Y \cap U_i \mid U_i \text{ is a definable open subset of } X \text{ and } Y \cap U_i \text{ is definable}\}.$$

(The above definition does not depend on any arbitrary choice of an admissible covering.) Local subsets are not called subspaces! Their use often deviates from their "inherited space structure" (even if they are definable), since we mainly want to consider "locally definable functions" on them. (A function is called here "locally definable" if its domain and codomain are local subsets of some objects of $\mathbf{LDS}(R)$, and all function germs of this function at points of its domain are definable. A function germ f_x at x is called definable if some definable neighborhood of x is mapped by f into a definable neighborhood of $f(x)$ and the obtained restriction of f is a definable mapping.)

Example 10.

a) The semialgebraic set $(0,1)_{\mathbb{R}}$ inherits an affine semialgebraic space structure from $\mathbb{R}$. Nevertheless, when speaking about "locally definable functions" (into $\mathbb{R}$) we want to treat it as a "localized" open interval $(0,1)_{loc}$, which is not a semialgebraic space.

b) Consider the semialgebraic set $S = (0,1)^2 \cup \{(1,1)\}$ in $\mathbb{R}^2$. The fact of being a "locally semialgebraic function" on S (into $\mathbb{R}$) does not reduce to being a morphism of any locally (and even weakly) semialgebraic space that can be formed by redefining the notion of admissible covering of the space S.

General definable spaces and locally definable spaces do not behave well enough for being used in homotopy theory. The right choice of assumptions are: regularity and one new called "paracompactness", which is only an analogue of the topological notion.

3 Locally definable spaces

One of the reasons why we pass to the locally definable spaces is the need of existence of covering mappings with infinite (for example countable) fibres.

Example 11 (cf. 5.14 in [7])**.** The space $\mathrm{Fin}(R)$. *We look for the* universal covering *of the unit circle $S^1 \subseteq R^2$. We will see soon that (as in topology) $\pi_1(S^1) = \mathbb{Z}$, so the universal covering should have countable fibres. Let $Fin(R)$ be the locally definable space introduced as the union of all intervals $(-n, n)$ in R for $n \in \mathbb{N}$. There is a surjective*

semialgebraic morphism $e : [0,1] \to S^1$ *that maps 0 and 1 to the distinguished point on* S^1 *and is injective elsewhere. Then the universal covering mapping* $p : Fin(R) \to S^1$ *defined by* $p(m+x) = e(x)$*, where* $m \in \mathbb{Z}, x \in [0,1]$*, is a morphism of locally definable spaces.*

A family of subsets of a locally definable space is **locally finite** if each open definable subset of the space meets only finitely many members of the family.

A locally definable space is called **paracompact** if there is a locally finite covering of the whole space by open definable subsets. (Such a covering must be admissible, since "admissible" means: when restricted to an open definable subspace, there is a finite subcovering. Shortly: "admissible" is "locally essentially finite".)

Remark 12. *The locally definable space* $Fin(R)$ *given by the admissible covering* $\{(-n,n) : n \in \mathbb{N}\}$ *is always paracompact. (Notice that if* R *contains* $\mathbb{R}$*, then* $\bigcup_{r \in \mathbb{R}_+} (-r,r) = \bigcup_{n \in \mathbb{N}} (-n,n)$*. In the language of nonstandard analysis, we can say that each galaxy may be considered as a regular paracompact locally definable space.)*

Direct (cartesian) products preserve regularity and paracompactness of locally definable spaces (cf. I.4.2c) and I.4.4e) in [8]). We will denote the category of regular paracompact locally definable spaces over R by $\mathbf{RPLDS}(R)$.

Example 13. *The spaces from the proofs of Theorems 8 and 9 are objects of* $\mathbf{RPLDS}(R)$.

Remark 14. *A* ***connected*** *(in the sense of generalized topology: the space cannot be decomposed into two open disjoint nonempty subspaces) regular paracompact locally definable space has a countable admissible covering by definable open subsets (so called* ***Lindelöf property*** *in [8]). If it has finite dimension* k*, then it can be embedded into the cartesian power* $Fin(R)^{2k+1}$*. This holds by embedding into a partially complete space, triangulation (see Theorems 24, 25 below) and Theorem 3.2.9 from a book of Spanier [24] (see also II.3.3 of [8]).*

Topological Remark 15. *The notion of paracompactness introduced above differs from the topological one. Each definable space is paracompact. There are Hausdorff definable (so paracompact) spaces which are not regular. With the regularity assumption, each paracompact space is normal and admits partition of unity. Paracompactness is inherited by all subspaces and cartesian products. The Lindelöf property gives paracompactness only with the assumption that the closure of a definable set is definable.*

Fiber products exist in the category of locally definable spaces over R (cf. I.3.5). A morphism $f : M \to N$ between locally definable spaces is called **proper** if it is universally closed in the sense of the generalized topology. This means that for each morphism of locally definable spaces $g : N' \to N$ the induced morphism $f' : M \times_N N' \to N'$ in the pullback diagram is a closed mapping in the sense of the generalized topological spaces (it maps closed subspaces onto closed subspaces). If all restrictions of f to closed definable subspaces are proper, then we call f **partially proper**.

A Hausdorff locally definable space M is called **complete** if the only morphism from M to the one point space is proper. Each paracompact complete space is affine definable (compare I.5.10 in [8]). Moreover, M is called **locally complete** if each point has a complete neighborhood. (Each locally complete locally definable space is regular, cf. I.7, p. 75 in [8]). It is **partially complete** if every closed definable subspace is complete. Every partially complete regular space is locally complete (cf. I.7.1 a)) in [8]).

Topological Remark 16. *This notion of properness is analogical to a notion from algebraic geometry. Partial completeness is the key notion.*

Let M be a locally complete paracompact space. Take the family $\overset{\circ}{\gamma}_c(M)$ of all such open definable subsets U of M that $\overline{U}$ is complete. Introduce a new locally definable space M_{loc}, the **localization** or **partial completetization** of M, on the same underlying set taking $\overset{\circ}{\gamma}_c(M)$ as an admissible covering by small open subspaces. The new space is regular partially complete (not only locally complete) and the identity mapping from M_{loc} to M is a morphism, but M_{loc} may not be paracompact, see Warning-Example 22. Notice that localization leaves the strong topology unchanged.

Topological Remark 17. *Localization is similar to the process of passing to k-spaces (they are the same as compactly generated spaces if hausdorffness is assumed) in homotopy theory. But notice that each topological locally compact space is a k-space.*

Remark 18. *Only one of the four locally semialgebraic spaces of Theorem 8 is partially complete, namely $\mathbb{R}_{loc}$.*

A **paracompact locally definable manifold** of dimension n over R is a Hausdorff locally definable space over R that has a locally finite covering by definable open subsets that are isomorphic to open balls in R^n. (Such a space is paracompact and locally complete, so regular, cf. I.7 p.75 in [8].) If additionally the transition maps are (definable) C^k-diffeomorphisms ($k = 1, ..., \infty$), then we get **paracompact locally definable C^k-manifolds**. Notice that the differential structure of such manifolds may be encoded in sheaves (in the sense of the strong topology) of C^k functions.

Proposition 19. *Paracompact analytic manifolds of dimension n are in bijective correspondence with partially complete paracompact locally definable C^∞-manifolds over $\mathbb{R}_{an}$ of the same dimension.*

Proof. A paracompact analytic manifold induces a paracompact locally definable C^∞-manifold over $\mathbb{R}_{an}$: Obviously, it is regular. We may assume (by shrinking the coverings if necessary) that the analytic structure of the manifold is given by a locally finite atlas consisting of charts whose domains and ranges are relatively compact subanalytic sets, and the charts extend analytically beyond the closures of chart domains. By taking a nice locally finite refinement, we additionally can get the chart domains and chart ranges (analytically and globally subanalytically) isomorphic to open balls in $\mathbb{R}^n$. Now the chart domains form a locally finite covering of the analytic manifold that defines a paracompact locally definable manifold over $\mathbb{R}_{an}$. The transition maps are $\mathbb{R}_{an}$-definable C^∞-diffeomorphisms of open, relatively compact, subanalytic subsets of some $\mathbb{R}^n$.

Notice that the relatively compact subanalytic sets are now the definable sets and the subanalytic sets are now the locally definable sets.

The obtained locally definable space is partially complete.

Vice versa: A paracompact locally definable C^∞-manifold over $\mathbb{R}_{an}$ induces a Hausdorff manifold with analytic globally subanalytic transition maps and globally subanalytic chart ranges. We may assume that the manifold is connected. Its locally finite atlas is countable (cf. I.4.17 in [8]), so it is a second countable topological space, and finally a paracompact analytic manifold. All locally definable subsets are now subanalytic (they are globally subanalytic in every chart).

One-to-one correspondence: If the paracompact locally definable manifold is partially complete, then: the definable subsets are exactly the relatively compact subanalytic subsets, and the locally definable subsets are exactly the subanalytic subsets of the obtained paracompact analytic manifold. Notice that the strong topology does not change when we pass from one type of a manifold to the other. So the structure of a partially complete locally definable space is uniquely determined (see Remark 7). Both the structures of a C^∞ locally definable manifold over $\mathbb{R}_{an}$ and the structure of an analytic manifold do not change during the above operations (only a convenient atlas was chosen). $\square$

Remark 20. *A real function on a (paracompact) analytic manifold M is analytic iff it is a C^∞ morphism from the corresponding partially complete paracompact locally definable C^∞-manifold (call it M_{ldm}) into $\mathbb{R}_{an}$. (See 5.3 in [9].)*

Analogously, for each expansion $\mathbb{R}_\mathcal{S}$ of the field $\mathbb{R}$ being a reduct of $\mathbb{R}_{an}$, partially complete paracompact locally definable C^∞-manifolds over $\mathbb{R}_\mathcal{S}$ correspond to some special kinds of paracompact analytic manifolds. Then the locally definable subsets in the sense of a given locally definable manifold as well as in the sense of its "expansions" (see below) form "nice geometric categories". This in particular generalizes the *analytic-geometric categories* of van den Dries and Miller [9].

The above phenomenon may be explained in the following way: The analytic manifolds $\mathbb{R}^n$ ($n \geq 1$), which model all analytic manifolds, have a natural notion of smallness. A subset $S \subset \mathbb{R}^n$ is **topologically small** if it is *bounded* or *relatively compact*. In the corresponding partially complete paracompact locally definable C^∞-manifolds $Fin(\mathbb{R}_{an})^n = (\mathbb{R}_{an})^n_{loc} = \bigcup_{k\in\mathbb{N}} (-k,k)^n = Fin((\mathbb{R}_{an})^n)$ over $\mathbb{R}_{an}$ this means that S is a *small* subset (if S is subanalytic then this means *definable* or *relatively complete*). It is partial completeness that gives analogy between usual topology and generalized topology.

Remark 21. *The generalized topology of the space M_{ldm} of Remark 20 is "the subanalytic site" considered by microlocal analysts (see [16]). More generally: the generalized topology of each paracompact locally definable manifold may be considered as a "locally definable site". It is also possible to consider all subanalytic subsets of a real analytic manifold as open sets of a generalized topological space. But then the strong topology becomes discrete.*

Warning-Example 22 (cf. I.2.6 in [8])**.** *(*The space R_{loc}.*) Our R is an affine definable space, which is locally complete but not complete. For such a space R_{loc} is introduced by the admissible covering $\{(-r,r) : r \in R_+\}$. This is a locally (even regular partially) complete space which is not definable. If the cofinality of R is uncountable, then R_{loc} is not paracompact! Here the morphisms from R to R are "the continuous definable functions", and the morphisms from R_{loc} to R are "the continuous locally (in the sense of R_{loc}) definable functions". (The latter case includes some nontrivial periodic funtions for an archimedean R.)*

A series of topological facts have counterparts for regular paracompact locally definable spaces.

Lemma 23 (cf. [7] and [8], chapter I)**.** *Let M be a regular paracompact locally definable space over R. Then:*

a) ***[closure property]*** *the closure of a definable set is definable (cf. I.4.6);*

b) ***[shrinking of coverings lemma]*** *for each locally finite covering (U_λ) of M by open locally definable sets there is a covering (V_λ) of M by open locally definable sets such that $\overline{V_\lambda} \subseteq U_\lambda$ (cf. I.4.11);*

c) ***[partition of unity]*** *for every locally finite covering (U_λ) of M by open locally definable subsets there is a subordinate partition of unity, i. e. there is a family of morphisms $\phi_\lambda : M \to [0,1]$ such that* $\mathrm{supp}\phi_\lambda \subseteq U_\lambda$ *and $\sum_\lambda \phi_\lambda = 1$ on M (cf. I.4.12);*

d) ***[Tietze's extension theorem]*** *if A is a closed subspace of M and $f : A \to K$ is a morphism into a convex definable subset K of R, then there exists a morphism $g : M \to K$ such that $g|A = f$ (cf. I.4.13);*

e) ***[Urysohn's lemma]*** *if A, B are disjoint closed locally definable subsets of M, then there is a morphism $f : M \to [0,1]$ with $f^{-1}(0) = A$ and $f^{-1}(1) = B$ (cf. I.4.15).*

Each locally definable space M over R has a natural "base field extension" $M(S)$ over any elementary extension S of R (cf. I.2.10 in [8]) and an "expansion" $M_{R'}$ to a locally definable space over any o-minimal expansion R' of R. Analogously, we may speak about a base field extension of a morphism.

The rules of conservation of the main properties under the base field extension are the same as for the locally semialgebraic case:

a) the base field extensions of the family of the connected components of a locally definable space M form the family of connected components of $M(S)$ (cf. I.3.22 i) in [8]);

b) if M is Hausdorff then: the space M is definable iff $M(S)$ is definable, M is affine definable iff $M(S)$ is affine definable, M is paracompact iff $M(S)$ is paracompact, M is regular and paracompact iff $M(S)$ is regular and paracompact (cf. B.1 in [8]).

c) if M is regular and paracompact, then: M is partially complete iff $M(S)$ is partially complete, M is complete iff $M(S)$ is complete (cf. B.2 in [8]).

If we expand R to an o-minimal R' then:

a) any locally definable space M is regular over R iff $M_{R'}$ is a regular space over R', since they have the same strong topologies;

b) a locally definable space M is connected over R iff $M_{R'}$ is connected over R' (for an affine space a clopen subset of a set definable over R is defibable over R, generally apply an admissible covering by affine subspaces "over R");

c) a locally definable space M is Lindelöf over R iff $M_{R'}$ is Lindelöf over R': if M is Lindelöf, then $M_{R'}$ is obviously Lindelöf; if $M_{R'}$ is Lindelöf then each member of a countable admissible covering $\mathcal{V}$ of $M_{R'}$ by definable open subspaces is covered by a finite union of elements of the admissible covering $\mathcal{U}$ of M by definable open subspaces that allowed to construct $M_{R'}$. Then $\mathcal{U}$ has a countable subcovering $\mathcal{U}'$. The family $\mathcal{U}''$ of finite unions of elements of $\mathcal{U}'$ is a countable coarsening of $\mathcal{V}$, hence is admissible in $M_{R'}$. Since "admissible" means "locally essentially finite", $\mathcal{U}''$ is also admissible in M. (The proof of Proposition 2.11 iii) in [2] does not care about admissibility);

d) if M is a Hausdorff locally definable space over R, then M is paracompact over R iff $M_{R'}$ is paracompact over R': if M is paracompact, then $M_{R'}$ is obviously paracompact; if $M_{R'}$ is paracompact, then we can assume that it is connected. Then $M_{R'}$ is Lindelöf (cf. I.4.17 in [8]) and **taut** (i.e. the closure of a definable set is definable, cf. I.4.6 in [8]). Now, by c), the space M is Lindelöf, and it is taut by the construction of $M_{R'}$ and considerations of Fundamental Example 1, so M is paracompact (see I.4.18 in [8] and Proposition 2.11 iv) in [2]).

4 Homotopies

Here basic definitions of homotopy theory are re-introduced.

Let M, N be objects of **Space**(R) and let f, g be morphisms from M to N. A **homotopy** from f to g is a morphism $H : M \times [0,1] \to N$ such that $H(\cdot, 0) = f$ and $H(\cdot, 1) = g$. If H exists, then f and g are called **homotopic**. If additionally $H(x,t)$ is independent of $t \in [0,1]$ for each x in a subspace A, then we say that f and g are **homotopic relative to** A. A subspace A of a space M is called a **retract** of M if there is a morphism $r : M \to A$ such that $r|A = id_A$. Such r is called a **retraction**. A subspace A of M is called a **strong deformation retract** of M if there is a homotopy $H : M \times [0,1] \to M$ such that H_0 is the identity and H_1 is a retraction from M to A. Then H is called **strong deformation retraction**.

A **system of spaces** over R is any tuple $(M, A_1, ..., A_k)$ where M is a space over R and $A_1, ..., A_k$ are subspaces of M. A **closed pair** is a system (M, A) of a space with a closed subspace. A system $(A_0, A_1, ..., A_k)$ is **decreasing** if A_{i+1} is a subspace of A_i for $i = 0, ..., k-1$. A **morphism** of systems of spaces $f : (M, A_1, ..., A_k) \to (N, B_1, ..., B_k)$ is a morphism of spaces $f : M \to N$ such that $f(A_i) \subseteq B_i$ for each $i = 1, ..., k$. A **homotopy** between two morphisms of systems of spaces f, g from $(M, A_1, ..., A_k)$ to $(N, B_1, ..., B_k)$ is a morphism

$$H : (M \times [0,1], A_1 \times [0,1], ..., A_k \times [0,1]) \to (N, B_1, ..., B_k)$$

with $H_0 = f$ and $H_1 = g$. The **homotopy class** of such a morphism f will be denoted by $[f]$ and the **set of all homotopy classes** of morphisms from $(M, A_1, ..., A_k)$ to $(N, B_1, ..., B_k)$ by

$$[(M, A_1, ..., A_k), (N, B_1, ..., B_k)].$$

If C is a closed subspace of M, and $h : C \to N$ is a pregiven morphism such that $h(C \cap A_i) \subseteq B_i$, then we denote the sets of classes of homotopy relative to C of mappings extending h by

$$[(M, A_1, ..., A_k), (N, B_1, ..., B_k)]^h.$$

Let us adopt the notation: $I = [0,1]$, $\partial I^n = I^n \setminus (0,1)^n$, and $J^{n-1} = \overline{\partial I^n \setminus (I^{n-1} \times \{0\})}$. For every pointed space (M, x_0) over R and $n \in \mathbb{N}^*$ we define the **(absolute) homotopy groups** as sets

$$\pi_n(M, x_0) = [(I^n, \partial I^n), (M, x_0)]$$

where the multiplication $[f] \cdot [g]$, for $n \geq 1$, is the homotopy class of

$$(f * g)(t_1, t_2, ..., t_n) = \begin{cases} f(2t_1, t_2, ..., t_n), 0 \leq t_1 \leq \frac{1}{2} \\ g(2t_1 - 1, t_2, ..., t_n), \frac{1}{2} \leq t_1 \leq 1. \end{cases}$$

For $n = 0$ we get (only) a set $\pi_0(M, x_0)$ of connected components of M with the base point the connected component of x_0. Also, as in topology, we define **relative homotopy groups**

$$\pi_n(M, A, x_0) = [(I^n, \partial I^n, J^{n-1}), (M, A, x_0)].$$

A morphism $f : M \to N$ is a **homotopy equivalence** if there is a morphism $g : N \to M$ such that $g \circ f$ is homotopic to id_M and $f \circ g$ is homotopic to id_N. We call $f : M \to N$ a **weak homotopy equivalence** if f induces bijections in homotopy sets ($\pi_0(\cdot)$) and group isomorphisms in all homotopy groups ($\pi_n(\cdot)$, $n \geq 1$). Analogously, we define homotopy equivalences and weak homotopy equivalences for systems of spaces.

The following topological operations may be not executable in the category of regular paracompact locally definable spaces over a given R. The **smash product** of two pointed spaces M, N is $M \wedge N = M \times N / M \vee N$, where $M \vee n$ denoted the wedge product of such spaces. The **reduced suspension** of M is $S^1 \wedge M$. The **mapping cylinder** of $f : M \to N$ is the space $M \times [0,1] \cup N$ with each point $(x, 1)$, $x \in M$, identified with $f(x)$. The **mapping cone** of f is the mapping cylinder of f divided by $M \times \{0\}$. The **cofiber** of $f : M \to N$ is the space $(([0,1] \times M) \cup_{1 \times M, f} N) / \{0\} \times M$.

5 Comparison theorems for locally definable spaces

In this section the two Comparison Theorems from [8] are extended, and the third is added. The first steps to do this are: embedding in a partially complete space and triangulation.

Theorem 24 (embedding into a partially complete space, cf. II.2.1)**.** *Each regular paracompact locally definable space over R is isomorphic to a dense locally definable subset of a partially complete regular paracompact space over R.*

Theorem 25 (triangulation, cf. II.4.4)**.** *Let M be a regular paracompact locally definable space over R. For a given locally finite family $\mathcal{A}$ of locally definable subsets of M, there is a simultaneous triangulation $\phi : X \to M$ of M and $\mathcal{A}$ (i. e. an isomorphism from the underlying set X, considered as a locally definable space, of a strictly locally finite (geometric) simplicial complex $(X, \Sigma(X))$ to M such that all members of $\mathcal{A}$ are unions of images of (open) simplices from $\Sigma(X)$).*

In particular, each object of **RPLDS**(R) is locally (pathwise) connected and even locally contractible.

As an illustration of the methods available by triangulation, the following Bertini or Lefschetz type theorem (known from complex algebraic and analytic geometry) is proven. (See [21] for a topological version.)

A subspace Δ of a locally definable space Y **nowhere disconnects** Y if for each connected open neighborhood W of any $y \in Y$ there is an open neighborhood $U \subseteq W$ of y such that $U \setminus \Delta$ is connected.

A morphism $p : E \to B$ in **LDS**(R) is a **branched covering** if there is a closed, nowhere dense **exceptional subspace** $\Delta \subseteq B$ such that $p|_{p^{-1}(B \setminus \Delta)} : p^{-1}(B \setminus \Delta) \to B \setminus \Delta$ is a **covering mapping** (this means: there is an admissible covering of $B \setminus \Delta$ by open subspaces, each of them well covered, analogically to the topological setting). If each of the **regular points** $b \in B \setminus \Delta$ of the branched covering $p : E \to B$ have the fiber of

the same cardinality, then this cardinality is called the **degree** of a branched covering $p : E \to B$.

Theorem 26 (cf. Thm. 1 in [21]). *Let Y be a simple connected (this assumes connected) object of* $\mathbf{RPLDS}(R)$*, Z be a connected, paracompact locally definable manifold over R of dimension at least 2, and $\pi : Y \times Z \to Y$ the canonical projection.*

Assume that $V \subset Y \times Z$ is a closed subspace such that the restriction $\pi_V : V \to Y$ is a branched covering of finite degree and an exceptional set Δ of this branched covering nowhere disconnects Y. Put $X = (Y \times Z) \setminus V$, and $L = \{p\} \times Z$, for some $p \in Y \setminus \Delta$.

If there is a morphism of locally definable spaces $h : Y \to Z$ over R with the graph contained in X, then the inclusion $i : L \setminus V \to X$ induces an epimorphism in the fundamental groups $i_* : \pi_1(L \setminus V) \to \pi_1(X)$.

Lemma 27 (straightening property, cf. Lemma 3 in [21]). *Every paracompact locally definable manifold M over R has the following straightening property:*

*For each set $J \subset [0,1] \times M$ such that the natural projection $\beta : [0,1] \times M \to [0,1]$ restricted to J is a covering mapping of finite degree, there exists an isomorphism, called the **straightening isomorphism**, $\tau : [0,1] \times M \to [0,1] \times M$ which satisfies the following three conditions:*

2.1) $\beta \circ \tau = \beta$,

2.2) $\tau \mid \{0\} \times M = id$,

2.3) $\tau(J) = [0,1] \times (\alpha(J \cap (\{0\} \times M)))$, where $\alpha : [0,1] \times M \to M$ is the natural projection.

Proof. Special case. Assume M is a unit open ball in R^m. The set J is a finite union of graphs of definable continuous mappings $[0,1]_R \to M$. We apply induction on the number n of these graphs.

If $n = 1$ then obviously the straightening exists (compare Lemma 2 in [21]), and the isomorphism may be chosen to extend to the identity on the unit sphere.

If $n > 1$ and the lemma is true for $n-1$, then we can assume that the first $n-1$ graphs are already straightened and that the images of the corresponding mappings (points $p_1, ..., p_{n-1}$) are not infinitely near to one another. Moreover, since the distance from the value of the last function $\gamma_n(t)$ to any of the distinguished points has a positive lower bound, we can assume $\gamma_n(t)$ is always outside some closed balls centered at p_i's with radius larger than some rational number. Now, we can cover the rest of the unit ball by finitely many regions that are each isomorphic to the open unit ball. Since the last function is definable, there is only finitely many transitions from one region to another when $t \in [0,1]_R$. We have the straightening inside each of the regions. By glueing such straightenings as in the proof of Lemma 3 of [21], we get the straightening of the whole n-th mapping. Again the straightening extends to the identity on the unit sphere.

General case. Since J is definable, it is contained in a finite union of open sets each isomorphic to the open unit ball in R^m. The thesis of the lemma extends by arguments similar to these of the special case. □

Proof of Theorem 26. Clearly, X is a connected and locally simply connected space. Let $j : L \setminus V \hookrightarrow X \setminus (\Delta \times Z)$ and $k : X \setminus (\Delta \times Z) \hookrightarrow X$ be the inclusions. Then the proof falls naturally into two parts.

Step 1. The induced mapping $j_* : \pi_1(L \setminus V) \to \pi_1(X \setminus (\Delta \times Z))$ is an epimorphism. (This step is analogous to Part 1 of the proof of Theorem 1 in [21].)

Let $u = (f, g) : [0, 1] \to X \setminus (\Delta \times Z)$ be any loop at (p,h(p)). We can define a new loop $w = (\tilde{f}, \tilde{g}) : [0, 1] \to X \setminus (\Delta \times Z)$ by the formula

$$w(t) = \begin{cases} (f(2t), g(2t)) & \text{for } 0 \le t \le \frac{1}{2}, \\ (f(2-2t), h(f(2-2t))) & \text{for } \frac{1}{2} < t \le 1. \end{cases}$$

Since Y is simply connected, we have $[w] = [u] \in \pi_1(X \setminus (\Delta \times Z))$. Let us define two mappings

$$A : [0, 1] \ni t \mapsto (t, \tilde{g}(t)) \in [0, 1] \times Z,$$

$$\Omega : [0, 1] \times Z \ni (t, z) \mapsto (\tilde{f}(t), z) \in Y \times Z.$$

The restriction $\beta \mid \Omega^{-1}(V)$ of the natural projection $\beta : [0, 1] \times Z \to [0, 1]$ to $\Omega^{-1}(V)$ is a covering of finite degree. By Lemma 27, it has a straightening isomorphism $\tau : [0, 1] \times Z \to [0, 1] \times Z$. Set $\hat{t} = \frac{1}{2} - \mid t - \frac{1}{2} \mid$ and $\tau_t = (\alpha \circ \tau)(t, \cdot) : Z \to Z$, where $\alpha : [0, 1] \times Z \to Z$ is the natural projection. We can assume that $\tau_t = \tau_{\hat{t}}$ because $\tilde{f}(t) = \tilde{f}(\hat{t})$. The homotopy $H(t, s) = (\tilde{f}(\hat{t}(1 - s)), (\tau^{-1}_{\hat{t}(1-s)} \circ \tau_{\hat{t}} \circ \tilde{g})(t))$ joins the loop $w = H(\cdot, 0)$ to the loop $H(\cdot, 1)$ whose image is in $L \setminus V$. Notice that $H(0, s) = H(1, s) = (p, h(p))$ for every $s \in [0, 1]$. This implies that $[u] = [H(\cdot, 1)] \in j_*(\pi_1(L \setminus V))$ and completes the proof of Step 1.

Step 2. The mapping $k_* : \pi_1(X \setminus (\Delta \times Z)) \to \pi_1(X)$ induced by k is an epimorphism.

Notice that $(\Delta \times Z) \cap X$ nowhere disconnects X. Let $u = (f, g)$ be a loop in X at $(p, h(p))$. The set $\mathrm{im}(u)$ has an affine open neighborhood W.

We use a (locally finite) triangulation "over $\mathbb{Q}$" of $Y \times Z$ (that is an isomorphism $\phi : X \to Y \times Z$ for some strictly locally finite, not necessary closed, simplicial complex $(X, \Sigma(X))$, keeping the notation of [8]), compatible with $\mathrm{im}(u), \Delta \times Z, V, L, h, W$.

There is $\varepsilon \in \mathbb{Q}$ such that the "distance" from $\phi^{-1}(\mathrm{im}(u))$ to $\phi^{-1}(V \cap W)$ is at least ε. Moreover, the "velocity" of $\phi^{-1} \circ u$ (existing almost everywhere) is bounded by some rational number. Since now everything is piecewise linear over $\mathbb{Q}$, the Lebesque number argument is available. By the use of barycentric coordinates and the "distance" function for the chosen triangulation, we find a loop $\tilde{u} = (\tilde{f}, \tilde{g})$ homotopic to u rel $\{0, 1\}$ with image in $X \setminus (\Delta \times Z)$. □

The following lemmas and theorems, whose proofs use the machinery of *good triangulations*, are straightforward generalizations of the corresponding semialgebraic versions from [8]:

Theorem 28 (canonical neighborhood retraction, cf. III.1.1)**.** *Let M be an object of* **RPLDS**(R) *and A a closed subspace. There is an open neighborhood U of A and a strong deformation retraction*

$$H : \overline{U} \times [0, 1] \to \overline{U}$$

from $\overline{U}$ to A such that the restriction $H|U \times [0, 1]$ is a strong deformation retraction from U to A.

Lemma 29 (extension of morphisms, cf. III.1.2)**.** *Let M be an object of* **RPLDS**(R)*, A a closed subspace, and U a neighborhood of A from the previous theorem. Any morphism $f : A \to Z$ into a regular paracompact locally definable space extends to a morphism $\tilde{f} : \overline{U} \to Z$. Moreover, if $\tilde{f}_1$, $\tilde{f}_2$ are extensions of f to $\overline{U}$, then they are homotopic in $\overline{U}$ relative to A.*

Lemma 30 (homotopy extension property, cf. III.1.4)**.** *Let M be an object of* $\mathbf{RPLDS}(R)$*. If A is a closed subspace of M, then $(A \times [0,1]) \cup (M \times \{0\})$ is a strong deformation retract of $M \times [0,1]$. In particular, the pair (M, A) has the following Homotopy Extension Property:*

for each morphism $g : M \to Z$ into a regular paracompact locally definable space Z and a homotopy $F : A \times [0,1] \to Z$ with $F_0 = g|A$ there exists a homotopy $G : M \times [0,1] \to Z$ with $G_0 = g$ and $G|A \times [0,1] = F$.

Since our spaces may be triangulated, the method of *simplicial approximations* (III.2.5 of [8]) makes a good job. In particular, the method of *well cored systems* and *canonical retractions* from III.2 in [8] gives the following

Fact 31. *Each object of* $\mathbf{RPLDS}(R)$ *is homotopy equivalent to a partially complete one. A system $(M, A_1, ..., A_k)$ of a regular paracompact locally definable space with closed subspaces is homotopy equivalent to an analogous system of partially complete spaces.*

The following two main theorems from [8] generalize, but the *mapping spaces* from III.3, which depend on the degrees of polynomials, should be replaced with similar mapping spaces depending on concrete formulas $\Psi(\overline{x}, \overline{y}, \overline{z})$ with parameters $\overline{z}$ (one "mapping space" per each formula Ψ).

Let $(M, A_1, ..., A_r)$ and $(N, B_1, ..., B_r)$ be systems of regular paracompact locally definable spaces over R, where each $A_i (i = 1, ..., r)$ is closed in M. Let $h : C \to N$ be a given morphism from a closed subspace C of M such that $h(C \cap A_i) \subseteq B_i$ for each $i = 1, ..., r$. Then we have

Theorem 32 (first comparison theorem, cf. III.4.2)**.** *Let $R \prec S$ be an elementary extension. Then the "base field extension" functor from R to S induces a bijection between the homotopy sets:*

$$\kappa : [(M, A_1, ..., A_k), (N, B_1, ..., B_k)]^h \to [(M, A_1, ..., A_k), (N, B_1, ..., B_k)]^h(S).$$

Theorem 33 (second comparison theorem, cf. III.5.1)**.** *Let R be an o-minimal expansion of $\mathbb{R}$. Then the "forgetful" functor* $\mathbf{RPLDS}(R) \to \mathbf{Top}$ *to the topological category induces a bijection between the homotopy sets*

$$\lambda : [(M, A_1, ..., A_k), (N, B_1, ..., B_k)]^h \to [(M, A_1, ..., A_k), (N, B_1, ..., B_k)]^h_{top}.$$

Moreover, a version of the proof of the first comparison theorem gives

Theorem 34 (third comparison theorem)**.** *If R' is an o-minimal expansion of R, then the "expansion" functor induces a bijection between the homotopy sets*

$$\mu : [(M, A_1, ..., A_k), (N, B_1, ..., B_k)]^h_R \to [(M, A_1, ..., A_k)_{R'}, (N, B_1, ..., B_k)_{R'}]^h_{R'}.$$

E. Baro and M. Otero [1] have written a detailed proof of this theorem in the case of systems of definable sets. They use a natural tool of "normal triangulations" to get an applicable version of II.4.3 from [8]. The theorem extends to the general case.

Because of the locally finite character of the regular paracompact locally definable spaces, by inspection of the proof of the triangulation theorem (II.4.4 in [8]), each such space has an isomorphic copy that is built from sets definable without parameters glued

together along sets that are definable without paramaters. It is possible to triangulate even "over the field of real algebraic numbers $\overline{\mathbb{Q}}$ " or "over the field of rational numbers $\mathbb{Q}$". Moreover, if two 0-definable subsets of R^n are isomorphic as definable spaces (i. e. definably homeomorphic), then there is a 0-definable isomorphism between them (we may change arbitrary parameters into 0-definable parameters in the defining formula of an isomorphism).

By the *o-minimal version of Hauptvermutung* for the structure R, we understand the following statement, which is a version of Question 1.3 in [3]:

Given two semialgebraic (definable in the field structure of R) sets in some R^n, if they are definably homeomorphic, then they are semialgebraically homeomorphic.

In other words: *if two affine semialgebraic spaces are isomorphic as definable spaces, then they are isomorphic as semialgebraic spaces.*

This condition does not depend on possible parameters from R, so the o-minimal Hauptvermutung is a property of the first order theory $Th(R)$ of R. Such theories may be called **HV-theories**. (Remember that M. Shiota proved in [23] that definably homeomorphic compact polyhedra in $\mathbb{R}^n$ are semilinearly homeomorphic.)

O-minimal Hauptvermutung case. If R is a model of an HV-theory, then the category of regular paracompact locally semialgebraic spaces over (the underlying field of) R may be viewed as a subcategory of $\mathbf{RPLDS}(R)$. Moreover, by triangulation with vertices having coordinates in the field of real algebraic numbers $\overline{\mathbb{Q}}$, we have the following fact:

Each regular paracompact locally definable space over R is isomorphic *to a regular paracompact locally semialgebraic space over (the underlying field of) R.*

In general different triangulations may lead to possibly non-isomorphic locally semialgebraic spaces but still the homotopy categories $\mathbf{HRPLSS}(R)$ and $\mathbf{HRPLDS}(R)$ of regular paracompact locally semialgebraic and locally definable spaces over the same R (forgetting taking a reduct) are equivalent (take the inclusion ("expansion") functor $\mathbf{HRPLSS}(R) \to \mathbf{HRPLDS}(R)$).

Analogously, by Theorem 25 and the Comparison Theorems, we get

Corollary 35. *The homotopy categories of: systems $(M, A_1, ..., A_k)$ of regular paracompact locally definable spaces with finitely many closed subspaces and systems $(M, A_1, ..., A_k)$ of regular paracompact locally semialgebraic spaces with finitely many closed subspaces (over the "same" R) are equivalent.*

Proof. By the triangulation theorem 25, every object of the former category is isomorphic to an object of the later category. Thus the "expansion" functor is essentially surjective. By the comparison theorem 34, it is also full and faithfull. This implies that this functor is an equivalence of categories. □

So the homotopy theory for regular paracompact locally definable spaces can (to a large extent) be transfered from the semialgebraic homotopy theory and, eventually, from the topological homotopy theory as in [8].

Other important facts about regular paracompact locally definable spaces will be developed in a more general setting of definable CW-complexes and weakly definable spaces.

6 Weakly definable spaces

In homotopy theory one needs to use quotient spaces (e. g. mapping cylinders, mapping cones, cofibers, smash products, reduced suspensions, CW-complexes), and this operation is not always executable in the category of locally definable spaces. That is why weakly definable spaces, which are analogues of arbitrary Hausdorff topological spaces, need to be introduced.

Let (M, O_M) be a space over R, and let K be a small subset of M. We can induce a space on K in the following way:

i) *open* sets in K are the intersections of open sets on M with K,

ii) *admissible* coverings in K are such open coverings that some finite subcovering already covers the union,

iii) a function $h : V \to R$ is a *section* of $O_K(V)$ if it is a finite open union of restrictions to K of sections of the sheaf O_M.

We call (K, O_K) a **small subspace** of (M, O_M).

A subset K of M is called **closed definable** in M if K is closed, small, and the space (K, O_K) is a definable space. The collection of closed definable subsets of M is denoted by $\overline{\gamma}(M)$. The set K is called a **polytope** if it is a closed definable complete space. We denote the collection of polytopes of M by $\gamma_c(M)$.

A **weakly definable space** (over R) is a space M (over R) having a family, indexed by a partially ordered set A, of (regular) closed definable subsets $(M_\alpha)_{\alpha \in A}$ such that the following conditions hold:

WD1) M is the union of all M_α,

WD2) if $\alpha \leq \beta$ then M_α is a (closed) subspace of M_β,

WD3) for each α there is only a finite number of β such that $\beta \leq \alpha$,

WD4) the family (M_α) is strongly inverse directed, i. e. for each α, β there is some γ such that $\gamma \leq \alpha$, $\gamma \leq \beta$ and $M_\gamma = M_\alpha \cap M_\beta$,

WD5) the set of indices is directed: for each α, β there is γ with $\gamma \geq \alpha$, $\gamma \geq \beta$,

WD6) the space M is the *inductive limit* of the spaces (M_α), what means the following:

a) a subset U of M is *open* iff each $U \cap M_\alpha$ is open in M_α,

b) an open family (U_λ) is *admissible* iff for each α the restricted family $(M_\alpha \cap U_\lambda)$ is admissible in M_α,

c) a function $h : U \to R$ on some open U is a *section* of O_M iff all the restrictions $h|U \cap M_\alpha$ are sections of respective sheaves O_{M_α}.

Such family (M_α) is called an **exhaustion** of M.

A space M is called a **weak polytope** if M has an exhaustion composed of polytopes. **Morphisms** and **isomorphisms** of weakly definable spaces are their morphisms and isomorphisms as spaces (we get the full subcategory **WDS**(R) of **Space**(R)).

A **weakly definable subset** is such a subset $X \subseteq M$ that: has definable intersections with all members of some exhaustion (M_α), and is considered with the exhaustion $(X \cap M_\alpha)$, hence it may be considered as a **subspace** of M.

A subset X of M is **definable** if it is weakly definable and the space (X, O_X) is definable. A subset X of M is definable iff it is weakly definable and is contained in a member of an exhaustion M_α (cf. IV.3.4).

The **strong topology** on M is the topology that makes the topological space M the respective inductive limit of the topological spaces M_α. The unpleasant fact about the weakly definable spaces (comparising with the locally definable spaces) is that points

may not have small neighborhoods (see Example 37). Moreover, the open sets from the generalized topology may not form a basis of the strong topology (cf. Appendix C in [17]).

The closure of a definable subset of M is always definable (cf. IV.3.6), so the topological closure operator restricted to the class $\gamma(M)$ of definable subsets of M may be treated as the closure operator of the generalized topology. The weakly definable subsets are "piecewise constructible" from the generalized topology.

All weakly definable spaces are Hausdorff, actually even "normal", see IV.3.12 in [17]. We can consider "expansions" and "base field extensions" of weakly definable spaces (compare considerations in IV.3) or morphisms (in the case of a base field extension) similar to the operations defined for locally definable spaces. They do not depend on the chosen exhaustion and preserve connectedness.

Fiber products exist in $\mathbf{WDS}(R)$ (cf. IV.3.20). So we (analogously to the case of locally definable spaces) define **proper** and **partially proper** mappings between weakly definable spaces as well as **complete** and **partially complete** spaces. It appears that the complete spaces are the polytopes, and the partially complete spaces are the weak polytopes (cf. IV.5 in [17]).

Example 36. *The category* $\mathbf{RPLDS}(R)$ *is a full subcategory of* $\mathbf{WDS}(R)$. *An exhaustion of an object* M *is given by all finite subcomplexes* Y *in* X *that are closed in* X *for some triangulation* $\phi : X \to M$ *(cf. IV.1.5).*

Examples 37. *An infinite wedge of circles is a weak polytope but not a locally definable space (cf. IV.1.8). A "countable comb" or "uncoutable comb" is a weak polytope which is not a locally definable space (cf. IV.4.7-8).*

Example 38. *Consider an uncountable subfield* F *of* $\mathbb{R}$. *Let* X *be a subset of the unit square* $[0,1]^2$ *consisting of points that have at least one coordinate in* F. *This set has a natural exhaustion making* X *into a weak polytope. This weak polytope is not locally simple connected.*

Example 39. *An open interval is a definable space but not a weak polytope, an infinite comb with such a "hand" is a weakly definable space but not a weak polytope.*

Warning-Example 40. *The topological closure of a weakly definable subset may not be weakly definable. Moreover, the naive "Arc Sellecting Lemma for weakly definable spaces" does not hold. (See IV.4.7.)*

Glueing weakly definable spaces is possible: for a closed pair (M, A) and a partially proper morphism $f : A \to N$ the quotient space of $M \sqcup N$ by an equivalence relation identifying each $a \in A$ with $f(a)$ is a weakly definable space $M \cup_f N$ called the **space obtained by glueing M to N along A by f**. Then the projection $\pi : M \sqcup N \to M \cup_f N$ is partially proper and strongly surjective, cf. IV.8.6. (A morphism $f : M \to N$ is **strongly surjective** if each definable subset of N is covered by the image of a definable subset of M.)

A family $\mathcal{A}$ of subsets of a weakly definable space M will be called **piecewise finite** if for each $D \in \gamma(M)$, the set D meets only finitely many members of $\mathcal{A}$. (Such families are called *partially finite* in [17].)

A **definable partition** of a weakly definable space M is a piecewise finite partition of M into a subset Σ of the family $\gamma(M)$, of definable subsets of M. An element τ of

Σ is an **immediate face** of σ if $\tau \cap (\overline{\sigma} \setminus \sigma) \neq \emptyset$. Then we write $\tau \prec \sigma$. A **face** of σ is an element of some finite chain of immediate faces finishing with σ. (Each σ has only finitely many immediate faces, even a finite numer of faces, cf. V.1.7).

A **patch decomposition** of M is a definable partition Σ of M such that: for each $\sigma \in \Sigma$ there is a number $n \in \mathbb{N}$ such that any chain $\tau_r \prec \tau_{r-1} \prec ... \prec \tau_0 = \sigma$ in Σ has length $r \leq n$. The smallest such n is called the **height** of σ and denoted by $h(\sigma)$. A **patch complex** is a pair $(M, \Sigma(M))$ consisting of a space M and a patch decomposition $\Sigma(M)$ of M. Elements of the patch decomposition are called **patches**.

Example 41. *Each exhaustion gives a patch decomposition of M (cf. V.1.4).*

Instead of triangulations in **RPLDS**(R), we have available in **WDS**(R) so called special patch decompositions. A **special patch decomposition** is such a patch decomposition that for each $\sigma \in \Sigma$, the pair $(\overline{\sigma}, \sigma)$ is isomorphic to the pair with the second element being a standard open simplex, and the first element this standard open simplex with some added open faces.

Theorem 42 (cf. V.1.12 in [17])**.** *Let M be an object of* **WDS**(R) *and let $\mathcal{A}$ be a piecewise finite family of subspaces. Then there is a simultaneous special patch decomposition of M and the family $\mathcal{A}$.*

A **relative patch decomposition** of a closed pair (M, A) is a patch decomposition Σ of the space $M \setminus A$. Then we denote by $\Sigma(n)$ the union of all patches of height n, by M_n the union of A and all $\Sigma(m)$ with $m \leq n$, $M(n)$ the "direct topological sum" of all closures $\overline{\sigma}$ where $\sigma \in \Sigma(n)$, and $\partial M(n)$ the direct sum of all frontiers $\partial\sigma = \overline{\sigma} \setminus \sigma$ of $\sigma \in \Sigma(n)$.

By $\psi_n : M(n) \to M_n$ we denote the union of all inclusions $\overline{\sigma} \to M_n$ with $\sigma \in \Sigma(n)$, and by $\phi_n : \partial M(n) \to M_{n-1}$ the restriction of ψ_n, which is called the **attaching map**. Then, since ϕ_n is partially proper (cf. VI.2 in [17]), we can express M_n as $M(n) \cup_{\phi_n} M_{n-1}$. The space M_n is called **n-chunk** and $M(n)$ is called **n-belt**. So each weakly definable space is built up by glueing direct topological sums of definable spaces to the earlier constructed spaces in countably many steps. In particular, definable versions of CW-complexes are among weakly definable spaces (see below).

A family $(X_\lambda | \lambda \in \Lambda)$ from $\mathcal{T}(M)$, the class of weakly definable subsets of M, is called **admissible** if each definable subspace B of M is contained in the union of finitely many elements of the family. (One could call such families "piecewise essentially finite".) Thus definable partitions are exactly the admissible partitions into definable subsets.

An **admissible filtration** of a space X is an admissible increasing sequence of closed subspaces $(X_n | n \in \mathbb{N})$ covering X. For example the sequence $(M_n | n \in \mathbb{N})$ of chunks of M (for a given patch decomposition) is an admissible filtration of M (cf. VI.2 in [17]).

The next lemma is very important in homotopy-theoretic considerations.

Lemma 43 (composition of homotopies, cf. V.5.1)**.** *Let $(C_n)_{n\in\mathbb{N}}$ be an admissible filtration of a space M. Assume $(G_n : M \times [0,1] \to N)_{n\in\mathbb{N}}$ is a family of homotopies such that $G_{n+1}(\cdot, 0) = G_n(\cdot, 1)$ and G_n is constant on C_n . For any given strictly increasing sequence $0 = s_0 < s_1 < s_2 < ...$ with all s_m less than 1 there is a homotopy $F : M \times [0,1] \to N$ such that*

$$F(x,t) = G_{k+1}(x, \frac{t - s_k}{s_{k+1} - s_k}), \text{ for } (x,t) \in C_n \times [s_k, s_{k+1}], 0 \leq k \leq n-2,$$

and $F(x,t) = G_n(x, 0)$ for $(x,t) \in C_n \times [s_{n-1}, 1]$.

7 Comparison theorems for weakly definable spaces

Now, with patch decompositions playing the role of triangulations we get the Comparison Theorems for weakly definable spaces as in [17].

Lemma 44 (homotopy extension property, cf. V.2.9)**.** *Let (M, A) be a closed pair of weakly definable spaces over R. Then $(A \times [0,1]) \cup (M \times \{0\})$ is a strong deformation retract of $M \times [0,1]$. In particular, the pair (M, A) has the following Homotopy Extension Property:*

for each morphism $g : M \to Z$ into a weakly definable space Z and a homotopy $F : A \times [0,1] \to Z$ with $F_0 = g|A$ there exists a homotopy $G : M \times [0,1] \to Z$ with $G_0 = g$ and $G|A \times [0,1] = F$.

Let $(M, A_1, ..., A_r)$ and $(N, B_1, ..., B_r)$ be systems of weakly definable spaces over R where each A_i is closed in M. Let $h : C \to N$ be a given morphism from a closed subspace C of M such that $h(C \cap A_i) \subseteq B_i$ for each $i = 1, ..., r$. Then we have

Theorem 45 (first comparison theorem, cf. V.5.2 i))**.** *For an elementary extension $R \prec S$ the following map, induced by the "base field extension" functor, is a bijection*

$$\kappa : [(M, A_1, ..., A_r), (N, B_1, ..., B_r)]^h \to [(M, A_1, ..., A_r), (N, B_1, ..., B_r)]^h(S).$$

Theorem 46 (second comparison theorem, cf. V.5.2 ii))**.** *If $R = \mathbb{R}$ as fields, then the following map, induced by the "forgetful" functor, to the topological homotopy sets is a bijection*

$$\lambda : [(M, A_1, ..., A_r), (N, B_1, ..., B_r)]^h \to [(M, A_1, ..., A_r), (N, B_1, ..., B_r)]^h_{top}.$$

Again, a version of the proof of the first comparison theorem (thus a version of the proof of V.5.2 i)) gives:

Theorem 47 (third comparison theorem)**.** *If R' is an o-minimal expansion of R, then the following map, induced by the "expansion" functor, is a bijection*

$$\mu : [(M, A_1, ..., A_k), (N, B_1, ..., B_k)]^h_R \to [(M, A_1, ..., A_k)_{R'}, (N, B_1, ..., B_k)_{R'}]^h_{R'}.$$

Proof. It suffices to prove the surjectivity, and only the case $k = 0$. We have a map $f : M \to N$ (over R') extending $h : C \to N$ (over R), and we seek for a mapping $g : M \to N$ (over R) such that g is homotopic to f relative to C (the homotopies appearing in this proof are allowed to be over R').

We choose a relative patch decomposition (over R) of (M, C), and will construct maps $h_n : M_n \to N$ (over R), $f_n : M \to N$ (over R') for $n \geq -1$, and a homotopy $H_n : M \times [0,1] \to N$ relative M_{n-1} such that: $h_{-1} = h$, $h_n|_{M_{n-1}} = h_{n-1}$, $f_{-1} = f$, $f_n|_{M_n} = h_n$, $H_n(\cdot, 0) = f_{n-1}$, $H_n(\cdot, 1) = f_n$. If we do this, we are done: we have a map $g : M \to N$ with $g|_{M_n} = h_n$ for each n. Composing, by Lemma 43, the homotopies $(H_n)_{n \geq 0}$ along a sequence $s_n \in [0,1)$ with $s_{-1} = 0$, we obtain a homotopy $G : M \times [0,1] \to N$ relative C from f to g as desired.

We start with $h_{-1} = h$, $f_{-1} = f$. Assume that h_i, f_i, H_i are given for $i < n$. Then we get a pushout diagram over R (see page 149 of [17]) and we define:

$$k_n = h_{n-1} \circ \phi_n : \partial M(n) \to N \text{ (over } R),$$

$$u_n = (f_{n-1}|_{M_n}) \circ \psi_n : M(n) \to N \text{ (over } R').$$

Notice that u_n extends k_n. By the comparison theorem for locally definable spaces (Theorem 34) there is a map $v_n : M(n) \to N$ over R extending k_n and a homotopy $F_n : M(n) \times [0,1] \to N$ relative $\partial M(n)$ from u_n to v_n. The maps v_n and h_{n-1} combine to a map $h_n : M_n \to N$, with $h_n \circ \psi_n = v_n$ and $h|_{M_{n-1}} = h_{n-1}$. The map F_n and $M_{n-1} \times [0,1] \ni (x,t) \mapsto h_{n-1}(x) \in N$ combine (cf. IV.8.7.ii) in [17]) to the homotopy $\widetilde{H}_n : M_n \times [0,1] \to N$ relative M_{n-1} from $f_{n-1}|_{M_n}$ to h_n. It can be extended (by Lemma 44) to the homotopy $H_n : M \times [0,1] \to N$ with $H_n(\cdot, 0) = f_{n-1}$. Put $f_n = H_n(\cdot, 1)$. This finishes the induction step and the proof of the theorem. □

O-minimal Hauptvermutung case.

If R is a model of an HV-theory, then the category of weakly semialgebraic spaces over (the underlying field of) R may be considered a subcategory of the weakly definable spaces over R. But see the following

Warning-Example 48. *Let Q be the square $[0,1]^2_R$. Now form $\widetilde{Q}$ by glueing $A \times S^1$ to Q by identifying $A \times \{1\}$ with A for each definable subset A of Q. If there are definable non-semialgebraic sets in R^2, then $\widetilde{Q}$ as a weakly definable space is not isomorphic to an expansion of a weakly semialgebraic space.*

8 Definable CW-complexes

A **relative definable CW-complex** (M, A) over R is a relative patch complex (M, A) satisfying the conditions:

(CW1) faces of patches have smaller dimensions than the original patches in the patch decomposition of $M \setminus A$,

(CW2) for each patch $\sigma \in \Sigma(M, A)$ there is $\chi_\sigma : E_n \to \overline{\sigma}$ (E_n denotes the unit closed ball) that maps the open ball isomorphically onto σ and the sphere onto $\partial\sigma$. For $A = \emptyset$, we have an **absolute definable CW-complex** over R. All definable CW-complexes are weak polytopes (absolute or relative, see V.7, p. 165, in [17]). A **system of CW-complexes** is a system of spaces $(M, A_1, ..., A_k)$ such that each A_i is a closed subcomplex of the CW-complex M.

Example 49. *Each partially complete object of* **RPLDS**(R) *is a definable CW-complex over R, since it is isomorphic to a closed (geometric) locally finite simplicial complex. (Compare considerations of II.4 and ii) in Examples V.7.1.)*

Fact 31 and Example 49 give

Fact 50. *Each object of* **RPLDS**(R) *is homotopy equivalent to a definable CW-complex over R. Each system $(M, A_1, ..., A_k)$ of a regular paracompact locally definable space with closed subspaces is homotopy equivalent to a system of definable CW-complexes.*

The following Whitehead theorem for definable CW-complexes may be proved like its topological analogue (see Theorem 7.5.4 in [20]).

Theorem 51. *Each weak homotopy equivalence between definable CW-complexes is a homotopy equivalence. Similar facts hold for any systems of definable CW-complexes.*

Proof. The proof is analogous to the proofs of 7.5.2, 7.5.3 and 7.5.4 in [20]. The argument from the long exact homotopy sequence may be proved like in [15] (compare III.6.1 in [8] and V.6.6 in [17]). The second part of the thesis follows immediately. □

Using the above instead of Theorem V.6.10 of [17], we get that

Theorem 52. *Each definable CW-complex is homotopy equivalent to an expansion of a base field extension of a semialgebraic CW-complex over $\overline{\mathbb{Q}}$. Analogous facts hold for decreasing systems of CW-complexes.*

Proof. This follows from the reasoning with relative CW-complexes analogous to the proof of V.7.10 in [17] (instead of the case of an elementary extension of real closed fields, we have the case of an o-minimal expansion of an o-minimal expansion of a real closed field). The construction of the desired relative CW-complex "skeleton by skeleton" is similar. Since we are dealing only with decreasing systems of CW-complexes, the use of V.6.10 of [17] (whose role is the transition from finite unions to any unions) may be replaced with the use of Theorem 51. □

Moreover, combining the above with the Comparison Theorems gives

Corollary 53. *The homotopy categories of: topological CW-complexes, semialgebraic CW-complexes over (the underlying field of) R, and definable CW-complexes over R are equivalent. Similar facts hold for decreasing systems of CW-complexes.*

9 The case of bounded o-minimal theories

Let T be an o-minimal complete theory extending RCF. We may assume that the theory is already Skolemized, so every 0-definable function is in the language and T has quantifier elimination. We can build models of T using the *definable closure* operation in some huge model (or, equivalently, using the notion of a *generated substructure* of a huge model for the chosen rich language). Taking a "primitive extension" generated by a single element gives a model of T determined up to isomorphism by the type this single element realizes over the former model.

Such a T will be called **bounded** if the model $P\langle t\rangle$ has countable cofinality, where P is the prime model of T and t realizes $+\infty$ over P. In other words: there is a sequence of 0-definable unary functions that is cofinal in the set of all 0-definable unary functions at $+\infty$ (this property does not depend on a model of T) . In particular, polynomially bounded theories are bounded. Notice that $P\langle t\rangle$ is cofinal in $R\langle t\rangle$, for any model R of T, if t realizes $+\infty$ over R.

Each bounded theory T has the following property: each model R has an elementary extension S such that both S and its "primitive extension" $S\langle t\rangle$, with t realizing $+\infty$ over S, have countable cofinality. (Take $S = R\langle t_1\rangle$, with t_1 realizing $+\infty$ over R).

Example 54. *Consider the closed m-dimensional simplex with one open face removed ($m \geq 2$), call this set A, as a definable subset of R^{m+1}. We want to introduce a partially complete space on the same set A. If R and $R\langle t\rangle$ have countable cofinality, then we can find a sequence of internal points tending to the barycenter of the removed face, and we can use a "cofinal at 0_+" sequence of unary functions tending to the zero function to produce an increasing sequence $(P_n)_{n\in\mathbb{N}}$ of polytopes covering our set A and such that any*

polytope contained in A is contained in some P_n. Then $(P_n)_{n\in\mathbb{N}}$ is an exhaustion of a weak polytope with the underlying set A. The old space and the new space on A have the same polytopes.(Compare the proof of Theorem IV.9.2.) A similar construction may be made if several open faces are removed.

The role of the boundedness assumption may be also seen by considering Example IV.9.12 in [17]. By the reasoning similar to that of V.7.8, we get

Theorem 55 (CW-approximation, cf. V.7.14)**.** *If T is bounded, then each decreasing system of weakly definable spaces $(M_0, ..., M_r)$ over R has a CW-approximation (that is a morphism $\phi : (P_0, ..., P_r) \to (M_0, ..., M_r)$ from a decreasing system definable CW-complexes such that every $\phi|_{P_i}$ is a homotopy equivalence).*

The methods to obtain this theorem include the use (as in IV.9-10 of [17]) of a so called **partially complete core** $P(M)$ **of a weakly definable space** M, which is an analogue and generalization of the localization M_{loc} for locally complete paracompact locally definable spaces M, and a **partially proper core** p_f **of a morphism** $f : M \to N$ of weakly definable spaces. (Note that it is sensible to ask for a partially complete core only if R has countable cofinality.) In particular, the strong Whitehead Theorem (cf. V.6.10), proved by methods of IV.9-10 and V.4.7, V.4.13 in [17], guarantees the extension of results to weakly definable spaces. Thus the homotopy category of decreasing systems of weakly definable spaces over R is equivalent to its full (homotopy) subcategory of decreasing systems of definable CW-complexes over R (see Theorem V.2.13 in [17]).

Corollary 56. *If T is bounded, then the homotopy categories of topological, semialgebraic and definable CW-complexes and of weakly definable spaces (over any model R of T) are equivalent. Similarly for decreasing systems of spaces.*

10 Generalized homology and cohomology theories

Now we have the operation of taking the (reduced) suspension $SM = S^1 \wedge M$ on the category of pointed weak polytopes $\mathcal{P}^*(R)$ over R, and on its homotopy category $H\mathcal{P}^*(R)$ (cf. VI.1 in [17]). This allows to define analogues of so called *complete generalized homology and cohomology theories* known from homotopy theory just as in VI.2 and VI.3 of [17]. (Such theories do not necessarily satisfy the dimension axiom.) The category of abelian groups is denoted by *Ab*. For a pair (M, A) of pointed weak polytopes, M/A will denote the quotient space M by a closed space A, with the distinguished point being the point obtained from A.

A **reduced cohomology theory** k^* over R is a sequence $(k^n | n \in \mathbb{Z})$ of contravariant functors $k^n : H\mathcal{P}^*(R) \to Ab$ together with natural equivalences $s^n : k^{n+1} \circ S \leftrightsquigarrow k^n$ such that the following hold:

Exactness axiom

For each $n \in \mathbb{Z}$ and each pair of pointed weak polytopes (M, A) the sequence

$$k^n(M/A) \xrightarrow{p^*} k^n(M) \xrightarrow{i^*} k^n(A)$$

is exact.

Wedge Axiom

For each $n \in \mathbb{Z}$ and each family $(M_\lambda | \lambda \in \Lambda)$ of pointed weak polytopes the mapping

$$(i_\lambda)^* : k^n(\bigvee_\lambda M_\lambda) \to \prod_\lambda k^n(M)$$

is an isomorphism.

A **reduced homology theory** h_* over R is a sequence $(h_n | n \in \mathbb{Z})$ of covariant functors $h_n : H\mathcal{P}^*(R) \to Ab$ together with natural equivalences $s^n : h_n \leftrightsquigarrow h_{n+1} \circ S$ such that the following hold:

Exactness axiom

For each $n \in \mathbb{Z}$ and each pair of pointed weak polytopes (M, A) the sequence

$$h_n(A) \xrightarrow{i_*} h_n(M) \xrightarrow{p_*} h_n(M/A)$$

is exact.

Wedge Axiom

For each $n \in \mathbb{Z}$ and each family $(M_\lambda | \lambda \in \Lambda)$ of pointed weak polytopes the mapping

$$(i_\lambda)_* : \bigoplus_\lambda h_n(M_\lambda) \to h_n(\bigvee_\lambda M_\lambda)$$

is an isomorphism.

Similarly, unreduced generalized homology and cohomology theories may be considered on the category $H\mathcal{P}(2, R)$ of pairs of weak polytopes.

If T is bounded, then all these generalized homology and cohomology functors can be built by using spectra for homology theories, or Ω-spectra for cohomology theories as in VI.8 of [17].

Corollary 57. *If T is bounded, then by the equivalency of respective homotopy categories of topological pointed CW-complexes and of pointed weak polytopes we get "essentially the same" homology and cohomology theories as the classical ones, known from topological homotopy theory.*

11 Open problems

The following problems are still open:

1) Can the assumption of boundedness of T in Theorem 55 and later be omitted? Is there a way of proving the strong Whitehead Theorem without methods of IV.9-10 of [17]?

2) Do the above consideratons lead to a "(closed) model category" (see [14], page 109, for the definition)? Such categories are desired in homotopy theory.

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